PMM U.S.S.R.,Vol.45,pp.672-676
0021-8928/82/5 0672$7.50/0

UDC 539.3:534.1

OPTIMAL DESIGN OF A SPHERICAL NET-LIKE SHELL WITH A FIXED FIRST EIGENFREQUENCY OF AXISYMMETRIC OSCILLATIONS*

V.A. PURTOV and G.I. PSHENICHNOV

A shell consisting of elastic bars the axes of which form a sufficiently dense net of equilateral triangles on a spherical surface (mean surface of the shell) is considered. A continuous computational model /1/ and the optimal control theory methods are used to obtain a relationship between the radius of the thin-walled, tubular transverse cross section of the bars and the coordinate along the meridian of the middle surface. The relationship is such, that the dimensionless parameter of the first eigenfrequency of the axisymmetric shell oscillation is equal to some given value, and the functional governing the shell material volume assumes a minimum value. Numerical computation shows that this leads to substantial saving of material compared with the design in which the transverse cross sections of the bars are kept constant.

1. Equations of state of the computational model of the shell with a net of bars under consideration, can be obtained as a particular case from the equations of work /1/. We shall write them in the form

$$N_1=\frac{E'h}{1-\nu_1^2}(\varepsilon_1+\nu_1\varepsilon_2),\quad N_2=\frac{E'h}{1-\nu_1^2}(\varepsilon_2+\nu_1\varepsilon_1),\qquad S=\frac{E'h}{2(1+\nu_1)}\omega,\quad H=\frac{E'h^3}{12(1+\nu_2)}\tau \tag{1.1}$$

$$M_1=-\frac{E'h^3}{12(1-\nu_2^2)}(\varkappa_1+\nu_2\varkappa_2),\quad M_2=-\frac{E'h^3}{12(1-\nu_2^2)}(\varkappa_2+\nu_2\varkappa_1)$$

$$E'=\frac{EF}{ah},\quad h=6\rho k_\gamma,\quad \nu_1=\frac{1}{3},\quad \nu_2=\frac{1-\gamma}{3+\gamma},\quad k_\gamma=\sqrt{\frac{1+\gamma}{3+\gamma}},\quad \rho=\sqrt{\frac{J_1}{F}},\quad \gamma=\frac{GJ_3}{EJ_1}$$

Here F, J_1 and J_3 denote the area and moments of inertia of the transverse cross section of the bars in bending mode in the plane normal to the middle surface of the shell, and under torsion, E and G are the Young's and elastic shear moduli of the bar material, and a is the height of the equilateral triangles of the net. In the case of a triangular net of bars, their flexural rigidity in the plane tangent to the middle surface can be neglected in practice ($J_2 \equiv 0$). When $\gamma=0$ (disregarding the torsional rigidity of the bars), the equations (1.1) are transformed into the corresponding formulas for isotropic shells with the Poisson's ratio of the material equal to $\nu={}^1/_3$. In what follows, we shall consider the bars of tubular, thin-walled cross section. Denoting by r and δ the radius and thickness of the bar walls and assuming that $\delta^\circ=\delta/a=\text{const}$, we obtain

$$E'=\frac{\sqrt{2}\pi\delta E}{3k_\gamma},\quad h=3\sqrt{2}k_\gamma r,\quad \rho=\frac{r}{\sqrt{2}}$$

The mass of the bars per unit surface of the shell middle surface is

$$3\mu F/a=\mu' h,\quad \mu'=6\pi\delta^\circ\mu/k_\gamma$$

where μ and μ' denote bar material density and relative density of the material of the computational model. The volume of the bars can be written in the form

$$I=2\sqrt{2}\pi^2\delta^\circ R^3 I^\circ \tag{1.2}$$

$$I^\circ=I^\circ(v)=\frac{1}{k_\gamma}\int_{\alpha_1}^{\alpha_2} v\sin\alpha\,d\alpha,\quad v=\frac{h}{R}$$

*Prikl.Matem.Mekhan.,45,No.5,895-901,1981 and we shall solve the problem using the optimal control theory methods.

2. Consider the following optimal control problem: to find a control function $v^*(t)$ and control parameters vector ξ^* ensuring the minimum of the functional

$$I(v,\xi)=\int_{t_0}^{T}F(t,x(t,v,\xi),v(t),\xi)\,dt+F_T(x(T,v,\xi),\xi)$$

on a fixed interval $[t_0, T]$, where x is an n-dimensional phase vector, v is a r-dimensional control function and ξ is the q-dimensional vector of control parameters. The variation of the phase vector is described by the following system of differential equations:

$$\dot{x}=f(t,x,v,\xi),\quad x(t_0)=x_0 \tag{2.1}$$

where a dot denotes differentiation with respect to t. The admissible set for x, v, ξ is determined by the system of constraints along the trajectory and terminal constraints

$$\begin{aligned}
&\Gamma^s(t,x,v,\xi)=0\quad (s=1,2,\ldots,l)\\
&\Gamma^s(t,x,v,\xi)\leqslant 0\quad (s=l+1,l+2,\ldots,m)\\
&\Gamma_T^s(x(T),v(T),\xi)=0\quad (s=1,2,\ldots,l_T)\\
&\Gamma_T^s(x(T),v(T),\xi)\leqslant 0\quad (s=l_T+1,l_T+2,\ldots m_T)
\end{aligned}$$

Use of the numerical methods presupposes one or another form of discretization of the initial problem. In particular, the method of reducing the optimal control problems to the problem of nonlinear programming (e.g. /2,3/) has found wide acceptance. We shall consider the case of intergrating the system (2.1) by the Euler method (a detailed account for an arbitrary Runge—Kutta scheme can be found in /4/).

The system can be replaced by the following scheme:

$$\begin{aligned}
&x_{i+1}=x_i+h_if_i(x_i,v_i,\xi)\ (i=1,2,\ldots,k-1)\\
&h_i=t_{i+1}-t_i,\ t_1=t_0,\ t_k=T,\ x_1=x_0
\end{aligned} \tag{2.2}$$

$v(t)=v(t_i)=v_i$ on the half-interval $[t_i, t_{i+1})$ (we shall denote, for brevity, any function $\varphi(t_i)$ by φ_i).

We write the system of constraints in the form $(i=1,2,\ldots,k-1)$:

$$\begin{aligned}
&\Gamma_i^s(x_i,v_i,\xi)=0\ (s=1,2,\ldots,l)\\
&\Gamma_i^s(x_i,v_i,\xi)\leqslant 0\ (s=l+1,l+2,\ldots,m)\\
&\Gamma_T^s(x_k,\xi)=0\ (s=1,2,\ldots,l_T)\\
&\Gamma_T^s(x_k,\xi)\leqslant 0\ (s=l_T+1,l_T+2,\ldots,m_T)
\end{aligned} \tag{2.3}$$

The minimized functional assumes the form (v is a vector of dimension kr):

$$I(x(v,\xi),v,\xi)=\sum_{i=1}^{k-1}h_iF_i[x_i(v_1,\ldots,v_{i-1},\xi),v_i,\xi]+F_T(x_k(v,\xi),\xi)$$

$$x=\{x_1^1,\ldots,x_1^n,\ldots,x_k^n\},\ v=\{v_1^1,\ldots,v_1^r,v_2^1,\ldots,v_2^r,\ldots,v_{k-1}^r\}$$

Thus the initial optimal control problem has been reduced to a problem of nonlinear programming, namely to the problem of finding a minimum of the functional $I(v,\xi)$ in the presence of the constraints (2.3).

A large class of the nonlinear programming methods will reduce the initial problem to that of solving a sequence of problems of unconditional minimization of a differential function of the form

$$G(x(v),v,\xi)=\sum_{i=1}^{k-1}h_iB_i(x_i,v_i,\xi)+b(x_k,\xi) \tag{2.4}$$

The method of external penalty functions in particular refers to such methods. Here the function G can be written in the form

$$G=I(x(v),v,\xi)+\tau\Big[\sum_{i=1}^{k-1}\Big[\sum_{s=1}^{l}(\Gamma_i^s)^2+\sum_{i=l+1}^{m}\varphi(\Gamma_i^s)\Big]+\sum_{s=1}^{l_T}(\Gamma_T^s)^2+\sum_{s=l_T+1}^{m}\varphi(\Gamma_T^s)\Big]$$

where τ is the penalty coefficient, $\varphi(g)$ is a function differentiable in g, $\varphi(g)>0$ and strictly increasing when $g > 0$, and $\varphi(g) = 0$ when $g \leqslant 0$.

Use of the gradient methods of unconditional minimization leads to the necessity of calculating the gradient of G. Computing the gradient by numerical methods requires the knowledge of the $\sim kr$ values of the function G (in real problems the quantity kr assumes, as a rule, the values ranging from 10^2 to 10^3). It follows that the trajectory must be computed the same number of times. An attempt at direct integration of the complicated function G over the components $(i = 1, 2, \ldots, k-1;\ j = 1, 2, \ldots, r)$, yields very bulky formulas. We shall follow a method given in /4/ enabling us to obtain the value of the gradient relatively simply.

We introduce a system of vectors p_i of dimension n, given by the recurrent relations

$$p_i = \frac{\partial G}{\partial x_i} + \frac{\partial x_{i+1}}{\partial x_i} p_{i+1} \tag{2.5}$$

$$p_k = \frac{\partial G}{\partial x_k} = \frac{\partial b(x_k, \xi)}{\partial x_k} \quad (i = 1, 2, \ldots, k-1)$$

We have for all v_i

$$\frac{dG}{dv_i} = \frac{\partial G}{\partial v_i} + \frac{\partial x_{i+1}}{\partial v_i} p_{i+1} \quad (i = 1, 2, \ldots, k-1) \tag{2.6}$$

Indeed, taking into account the relations (2.2) and using the rules of differentiating a function of a function, we obtain

$$\frac{dG}{dv_{k-1}} = \frac{\partial G}{\partial v_{k-1}} + \frac{\partial x_k}{\partial v_{k-1}} \frac{\partial G}{\partial x_k} = \frac{\partial G}{\partial v_{k-1}} + \frac{\partial x_k}{\partial v_{k-1}} p_k$$

$$\frac{dG}{dv_{k-2}} = \frac{\partial G}{\partial v_{k-2}} + \frac{\partial x_{k-1}}{\partial v_{k-2}} \frac{\partial G}{\partial x_{k-1}} + \frac{\partial x_{k-1}}{\partial v_{k-2}} \frac{\partial x_k}{\partial x_{k-1}} p_k = \frac{\partial G}{\partial v_{k-2}} + \frac{\partial x_{k-1}}{\partial v_{k-2}} p_{k-1}$$

Further, using the sequence (2.5), we obtain (2.6).

Similarly we derive the formula

$$\frac{dG}{d\xi} = \frac{\partial G}{\partial \xi} + \sum_{i=1}^{k} \frac{\partial x_i}{\partial \xi} p_i \tag{2.7}$$

Let us write the relations (2.5)—(2.7), obtained for the function G written in the form (2.4)

$$p_i = p_{i+1} + h_i \left[\frac{\partial B_i(x_i, v_i, \xi)}{\partial x_i} + \frac{\partial f_i(x_i, v_i, \xi)}{\partial x_i} p_{i+1} \right] \tag{2.8}$$

$$p_k = \frac{\partial b(x_k, \xi)}{\partial x_k} \quad (i = 1, 2, \ldots, k-1)$$

$$\frac{\partial G}{d\xi} = \frac{\partial x_1}{\partial \xi} p_1 + \sum_{i=1}^{k-1} h_i \left[\frac{\partial B_i(x_i, v_i, \xi)}{\partial \xi} + \frac{\partial f_i(x_i, v_i, \xi)}{\partial \xi} p_{i+1} \right] + \frac{\partial b(x_k, \xi)}{\partial \xi} \tag{2.9}$$

$$\frac{dG}{dv_i} = h_i \left[\frac{\partial B_i(x_i, v_i, \xi)}{\partial v_i} + \frac{\partial f_i(x_i, v_i, \xi)}{\partial v_i} p_{i+1} \right] \tag{2.10}$$

Thus we see that to obtain the gradient of G, we must integrate the system (2.2), calculate p_i from the recurrence relations (2.8), and substitute them into the formulas (2.9), (2.10).

Let us now return to the optimization problem in question. Using (1.1) we can write the system of differential equations of free axisymmetric oscillations of the net-like spherical shell of radius R, in the form

$$\begin{aligned}
x^{\cdot 1} &= -(1 - \nu_1)x^1 \operatorname{ctg} t - x^2 + vx^4 \operatorname{ctg} t + (\operatorname{ctg}^2 t - \Omega^2)\, vx^5 \\
x^{\cdot 2} &= (1 + \nu_1)x^1 - x^2 \operatorname{ctg} t + (1 - \Omega^2)\, vx^4 + vx^5 \operatorname{ctg} t \\
x^{\cdot 3} &= -x^2 - (1 - \nu_2)x^3 \operatorname{ctg} t - v^3 \operatorname{ctg}^2 t x^6/12 \\
x^{\cdot 4} &= x^5 - x^6 \\
x^{\cdot 5} &= (1 - \nu_1^2)v^{-1}x^1 - (1 + \nu_1)x^4 - \nu_1 x^5 \operatorname{ctg} t \\
x^{\cdot 6} &= -12\,(1 - \nu_2^2)v^{-3}x^3 - \nu_2 x^6 \operatorname{ctg} t \\
x^1 &= N_1/ER, \quad x^2 = Q/ER, \quad x^3 = M_1/ER^2 \\
x^4 &= w/R, \quad x^5 = u/R, \quad x^6 = -\gamma_1
\end{aligned} \tag{2.11}$$

($\alpha = t$ is the angle of width of the middle surface). For the dimensionless parameter of the circular frequency Ω of free oscillations, we obtain

$$\Omega^2 = \frac{\mu' R^2 \omega^2}{E'} = \frac{3\mu R^2 \omega^2}{E}$$

Let us assume that the shell is rigidly clamped at the mutually parallel $t = t_0$ and $t = T$

$$x^4 = x^5 = x^6 = 0 \text{ for } t = t_0, \ T \tag{2.12}$$

The problem is formulated as follows: to find, for a fixed value of the dimensionless parameter of the first eigenfrequency of the natural axisymmetric oscillations of the spherical shell, the control function $v^*(t)$ ensuring a minimum of the functional (1.2) (minimum volume of the shell bars). Moreover, the phase vector must satisfy the differential equations (2.11) and constraints (2.12).

3. We give the results of solving the problem for the case

$$t_0 = \pi / 6; \quad T = \pi / 2; \quad \Omega^2 = 1.28$$

The Euler scheme with recomputation at $k = 102$ ensures a sufficient accuracy of computations. Since the solution of the homogeneous boundary value problem (2.11), (2.12) is obtained with the accuracy of up to the constant multiplier, we put $x^1(t_0) = 1$ (normalization of the eigenfunctions) and thus discard the trivial solution. We minimize the function

$$k_\gamma I^\circ = \sum_{i=1}^{k-1} h_i v_i \sin t_i$$

in the presence of the terminal constraints

$$\Gamma_T{}^1 = x^4(T) = 0, \quad \Gamma_T{}^2 = x^5(T) = 0, \quad \Gamma_T{}^3 = x^6(T) = 0$$

with the constraints along the trajectory absent. The function (2.4) assumes the form

$$G = \sum_{i=1}^{k-1} h_i v_i \sin t_i + \tau\, [(\Gamma_T{}^1)^2 + (\Gamma_T{}^2)^2 + (\Gamma_T{}^3)^2]$$

and the control parameters vector is $\xi = (x^2(t_0), x^3(t_0))$. The method of penalty functions was used to obtain the initial approximation v_0, ξ_0, and the solution v^*, ξ^* was then found with prescribed accuracy using the method of generalized Lagrange multipliers. The reasons for this choice of methods and the order in which they were used the following. The penalty function method discovers, relatively rapidly, the region which we shall describe as good initial approximation. The attempt, however, to obtain the exact value of the extremum leads to the appearance of characteristic "rolling", when the value of the penalty decreases appreciably from one cycle to the next while the value of the function increases, and vice versa. During this process the value of G decreases by an insignificant amount. On the other hand, is known that the method of Lagrange multipliers is preferably used when the initial point lies at some distance from the extremum.

For the unconditional minimization the method of conjugate gradients with a small reduction cycle, was found to be the most suitable, since the undulating character of the function G leads to appreciable errors in selecting the direction in the course of increasing the parameter of the reduction cycle. For this reason the method of conjugate gradients was found to be more effective than the method of quickest descent or the methods not utilizing the gradient (such as the Hooke-Jeeves method).

In the case $\gamma = 0$ (torsional rigidity of the bars is disregarded) the change in the dimensionless radius of the tubular cross section of the bars along the meridian of the shell middle surface, is characterized by the solid curve in Fig.1 and we have here $I^\circ = 5.41 \cdot 10^{-2}$. A solution for $v = \text{const}$ was obtained for comparison. In this case we assumed that $\xi = (x^2(t_0), x^3(t_0), v)$, and we minimized the error function at the right end of the trajectory

$$\min_{\xi} [(x^4(T))^2 + (x^5(T))^2 + (x^6(T))^2]$$

The method of conjugate gradients was used to obtain the solution. Since the system of equations (2.11) is linear, we have also obtained a solution based on the orthogonal run. The result obtained in the second case was found to be slightly better since it used less computer time. It follows therefore that the error function is best minimized with nonlinear equations.

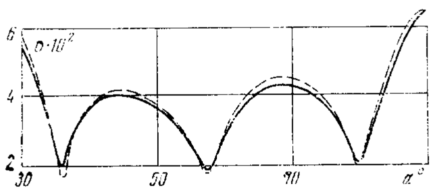

Fig.1

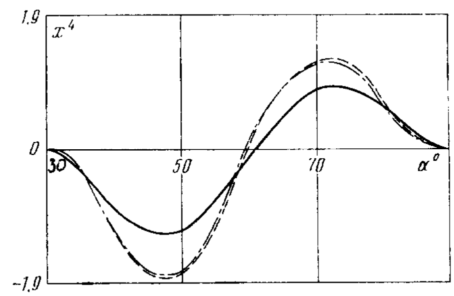

Fig.2

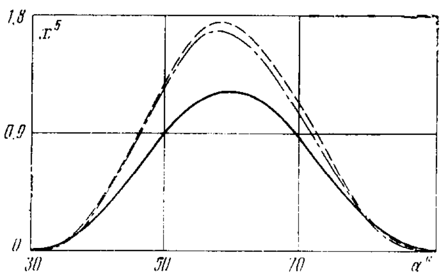

Fig.3

Solving the problem ($\gamma = 0$) gave v = 0.0450 (horizontal straight line in Fig.1), which corresponds to the value $I^\circ = 6.75 \cdot 10^{-2}$ of the functional. It follows that the volume of the shell bar material is in this case greater than the optimal volume, by 24.6%.

Taking into account the torsional rigidity of the bars and assuming that γ =0.769, we obtain a slightly different solution (dashed line in Fig.1). In this case we have $I^\circ = 4.72 \cdot 10^{-2}$ for the optimal solution (the shell bar material volume is reduced by 12.7% compared with the optimal variant of the net-like shell at $\gamma = 0$).

Figs.2 and 3 show the deflections and longitudinal displacements of the points of the middle surface of the shell. The dash and dot-dash curves correspond to the optimal variants for $\gamma = 0$ and $\gamma = 0.769$. Solid lines correspond to the case when $\gamma = 0$ and the radii of the tubular cross sections of the shell bars are constant.

REFERENCES

1. PSHENICHNOV G.I., Computation of net-like shells. In book: Investigations in the Theory of Structures. Ed.22, Moscow, STROIIZDAT, 1976.
2. MOISEEV N.N., Elements of the Theory of Optimal Systems. Moscow, NAUKA, 1975.
3. POLAK E., Numerical Optimization Methods. Moscow, MIR, , 1974.
4. GRACHEV N.I. and EVTUSHENKO Iu.G., Packet of programs for solving the optimal control problems. Moscow, VTs Akad. Nauk SSSR, 1978.

Translated by L.K.